Tentamen Signal Analysis, 29/10/03, room 11.0080, 9.00-12.00

Please write your name and student number on each sheet; answer either in Dutch or English.

Question 1

a. The function $h(t)$ is the convolution of the functions $f(t) = -2te^{-t^2}$ and $g(t) = e^{-t}$ for $t \geq 0$ and $g(t) = 0$ for $t < 0$. Sketch the functions *f* and *g*, explain in a graph how *f* and *g* are convolved, and sketch the resulting $h(t)$.
b. Show formally that $\mathcal{F}\{k'(t)\} = 2pif\,\mathcal{F}\{k(t)\}$, with $\mathcal{F}$ denoting a Fourier transformation, and $k'(t)$ is the time derivative of a function $k(t)$.
c. Calculate the Fourier transform of the functions $f(t)$, $g(t)$, and $h(t)$ defined in **a**.
(You may want to use $\mathcal{F}\{e^{-t^2}\} = \sqrt{p}\,e^{-(pf)^2}$)

Question 2
a. Define the Dirac delta function $d(t)$ as the limit of a rectangular pulse, and show, using that definition, that $x(t) = \int_{-\infty}^{\infty} x(t)d(t-t)dt$ for arbitrary functions $x(t)$.
b. A linear time-invariant system has a response $h(t)$ to a delta function as input. Show that the response $y(t)$ to an arbitrary input function $x(t)$ can be written as a convolution.
c. The impulse response of a linear time-invariant system is $h(t) = te^{-t/t}$ for $t \geq 0$ and $h(t) = 0$ for $t < 0$. Calculate the output $y(t)$ to an input signal $x(t) = \cos(2pf_1t) + \sin(2pf_2t)$. Hint: you may calculate this via the frequency domain; simplifying the resulting complex exponentials to real functions is not necessary.
d. The input to the system defined in **c** is now a stochastic signal $x(t)$ with power spectral density $S_{xx}(f) = d(f - f_1) + d(f - f_2)$. Calculate the power spectrum of the output.

Question 3
a. A random variable *X* has a probability density function (pdf) $p_X(x) = 1/p$ for $-p/2 \leq x \leq p/2$ and $p_X(x) = 0$ elsewhere. Draw the cumulative distribution function of *X*, and calculate the variance of *X*.
b. The random variable *X* is transformed into a random variable $Y = f(X)$. Show, using the cumulative distribution functions of *X* and *Y*, that $p_Y(y)dy = p_X(x)dx$ for a monotonously increasing function $f(X)$, and that $p_Y(y)dy = -p_X(x)dx$ for a monotonously decreasing function $f(X)$. Note that this implies $p_Y(y) = \left|\frac{dx}{dy}\right| p_X(x)$.
c. If $Y = \cos(X)$, calculate and draw the pdf of *Y*, given the pdf of *X* as defined in **a**.

Question 4
A stationary stochastic signal $x(t)$ is embedded in additive noise $n(t)$: the measured signal $y(t)$ is given by $y(t) = x(t) + n(t)$. The filter that optimally (in the least-squares sense) recovers the signal $x(t)$ from the measured signal is known as the Wiener filter, given by $H(f) = S_{xy}(f)/S_{yy}(f)$, with $S_{xy}(f)$ the cross spectral density of *x* and *y*, and $S_{yy}(f)$ the power spectral density of *y*. Suppose that the cross correlation of *x* and *n* $R_{xn}(t) = 0$, then find the Wiener filter if the autocorrelations of *x* and *n* are given by $R_{xx}(t) = Ae^{-t^2}$ and $R_{nn}(t) = Nd(t)$, with *A* and *N* constants. Calculate the power spectral density of the output of this Wiener filter when the input is the measured signal, and discuss for which frequencies it produces a good estimate of the power spectrum of *x*.